COMPLEX-VALUED SOLUTIONS OF THE BENJAMIN–ONO EQUATION

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Abstract. We prove that the Benjamin–Ono initial-value problem is locally well-posed for small data in the Banach spaces $\widetilde{H}^\sigma(\mathbb{R})$, $\sigma \geq 0$, of *complex-valued* Sobolev functions with special low-frequency structure.

1. Introduction

In this paper we consider the Benjamin–Ono initial-value problem

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u + \partial_x(u^2/2) = 0; \\ u(0) = \phi, \end{cases} \tag{1.1}$$

where $\mathcal{H}$ is the Hilbert transform operator defined by the Fourier multiplier $-i\,\mathrm{sgn}(\xi)$. This initial-value problem has been studied extensively for real-valued data in the Sobolev spaces $H^\sigma(\mathbb{R})$, $\sigma \geq 0$ (see, for example, the introduction of [7] for more references). In this paper we consider small *complex-valued* data with special low-frequency structure.

We define first the Banach spaces $\widetilde{H}^\sigma(\mathbb{R})$, $\sigma \geq 0$. Let $\eta_0 : \mathbb{R} \to [0,1]$ denote an even smooth function supported in $[-8/5, 8/5]$ and equal to 1 in $[-5/4, 5/4]$. For $l \in \mathbb{Z}$ let $\chi_l(\xi) = \eta_0(\xi/2^l) - \eta_0(\xi/2^{l-1})$, χ_l supported in $\{\xi : |\xi| \in [(5/8)\cdot 2^l, (8/5)\cdot 2^l]\}$. For simplicity of notation, we also define the functions $\eta_l : \mathbb{R} \to [0,1]$, $l \in \mathbb{Z}$, by $\eta_l = \chi_l$ if $l \geq 1$ and $\eta_l \equiv 0$ if $l \leq -1$. We define the Banach space $B_0(\mathbb{R})$ by

$$\begin{split} B_0 =& \{f \in L^2(\mathbb{R}) : f \text{ supported in } [-2,2] \text{ and} \\ & ||f||_{B_0} := \inf_{f=g+h} ||\mathcal{F}_{(1)}^{-1}(g)||_{L^1_x} + \sum_{k'=-\infty}^{1} 2^{-k'/2} ||\chi_{k'} \cdot h||_{L^2_\xi} < \infty\}. \end{split} \tag{1.2}$$

Here, and in the rest of the paper, $\mathcal{F}_{(d)}$ and $\mathcal{F}_{(d)}^{-1}$ denote the Fourier transform and the inverse Fourier transform on $\mathbb{R}^d$, $d = 1, 2$. For $\sigma \geq 0$ we define

$$\widetilde{H}^\sigma = \Big\{\phi \in L^2(\mathbb{R}) : ||\phi||_{\widetilde{H}^\sigma}^2 := ||\eta_0 \cdot \mathcal{F}_{(1)}(\phi)||_{B_0}^2 + \sum_{k=1}^{\infty} 2^{2\sigma k} ||\eta_k \cdot \mathcal{F}_{(1)}(\phi)||_{L^2}^2 < \infty\Big\}. \tag{1.3}$$

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The definition shows easily that $\widetilde{H}^\sigma \hookrightarrow H^\sigma$, $\sigma \geq 0$, and

$$\|\delta_\lambda(\phi)\|_{\widetilde{H}^\sigma} \leq C\|\phi\|_{\widetilde{H}^\sigma} \text{ for any } \lambda \in (0,1] \text{ and } \sigma \geq 0, \tag{1.4}$$

where $\delta_\lambda(\phi)(x) := \lambda\phi(\lambda x)$[1]. For any Banach space V and $r > 0$ let $B(r, V)$ denote the open ball $\{v \in V : ||v||_V < r\}$. Let $\mathbb{Z}_+ = \{0, 1, \dots\}$ and

$$\widetilde{H}^\infty = \bigcap_{\sigma \in \mathbb{Z}_+} \widetilde{H}^\sigma \text{ with the induced metric.}$$

Our main theorem concerns local well-posedness of the Benjamin–Ono initial-value problem (1.1) for small data in $\widetilde{H}^\sigma$, $\sigma \geq 0$.

Theorem 1.1. *(a) There is a constant $\overline{\epsilon} > 0$ with the property that for any $\phi \in B(\overline{\epsilon}, \widetilde{H}^0) \cap \widetilde{H}^\infty$ there is a unique solution*

$$u = S^\infty(\phi) \in C([-1,1] : \widetilde{H}^\infty)$$

of the initial-value problem (1.1)*.*

(b) The mapping $\phi \to S^\infty(\phi)$ extends (uniquely) to a Lipschitz mapping

$$S^0 : B(\overline{\epsilon}, \widetilde{H}^0) \to C([-1,1] : \widetilde{H}^0),$$

with the property that $S^0(\phi)$ is a solution of the initial-value problem (1.1) *for any $\phi \in B(\overline{\epsilon}, \widetilde{H}^0)$ (in the sense of distributions).*

(c) Moreover, for any $\sigma \in [0, \infty)$ we have the local Lipschitz bound

$$\sup_{t \in [-1,1]} \|S^0(\phi)(t) - S^0(\phi')(t)\|_{\widetilde{H}^\sigma} \leq C(\sigma, R)\|\phi - \phi'\|_{\widetilde{H}^\sigma} \tag{1.5}$$

for any $R > 0$ and $\phi, \phi' \in B(\overline{\epsilon}, \widetilde{H}^0) \cap B(R, \widetilde{H}^\sigma)$. As a consequence, the mapping S^0 restricts to a locally Lipschitz mapping

$$S^\sigma : B(\overline{\epsilon}, \widetilde{H}^0) \cap \widetilde{H}^\sigma \to C([-1,1] : \widetilde{H}^\sigma).$$

We discuss now some of the ingredients in the proof of Theorem 1.1. The main obstruction to proving a well-posedness result for the Benjamin–Ono equation using a fixed-point argument in some $X^{\sigma,b}$ space (in a way similar to the case of the KdV equation, see [2]) is the lack of control of the interaction between very high and very low frequencies of solutions (cf. [15] and [14]). The point of the low-frequency assumption $\eta_0 \cdot \widehat{\phi} \in B_0$ is to weaken this interaction.[2] Even with this low-frequency assumption, the use of standard $X^{\sigma,b}$ spaces for high-frequency functions (i.e. spaces defined by suitably weighted norms in the frequency space) seems to lead inevitably to logarithmic divergences (see [3] and section 5). To

[1]The inequality (1.4) does not improve, however, as $\lambda \to 0$, so the spaces $\widetilde{H}^\sigma$ are, in some sense, critical. Because of this we can only allow small data.

[2]Herr [5] has recently used spaces similar to $\widetilde{H}^\sigma$ to prove local and global well-posedness for the "dispersion-generalized" Benjamin–Ono equation, in which the term $\mathcal{H}\partial_x^2$ is replaced by $D_x^{1+\alpha}\partial_x$, $\alpha \in (0,1]$. In this case the logarithmic divergence mentioned above does not occur.

avoid these logarithmic divergences we work with high-frequency spaces that have two components: an $X^{\sigma,b}$-type component measured in the frequency space and a normalized $L^1_x L^2_t$ component measured in the physical space. This type of spaces have been used in the context of wave maps (see, for example, [11], [12], [22], [23], [24], [19], and [20]). Then we prove suitable linear and bilinear estimates in these spaces and conclude the proof of Theorem 1.1 using a recursive (perturbative) construction. Many of the estimates used in the proof of Theorem 1.1 have already been proved in [7]. There are, however, several technical difficulties due to the critical definitions of the spaces B_0 and $\widetilde{H}^\sigma$ (see (1.4)), which in this paper are larger than the corresponding spaces B_0 and $\widetilde{H}^\sigma$ in [7].

The rest of the paper is organized as follows: in section 2 we construct our main normed spaces and summarize some of their basic properties. In section 3 we state our main linear and bilinear estimates; most of these estimates, with the exception of Lemma 3.3, are already proved in [7]. In section 4 we combine these estimates and a recursive argument (in which we think of the nonlinear term as a perturbation) to prove Theorem 1.1. Finally, in section 5 we construct two examples that justify some of the choices we make in our definitions.

2. The normed spaces

Assume η_l and χ_l, $l \in \mathbb{Z}$, are defined as in section 1. For $l_1 \leq l_2 \in \mathbb{Z}$ let

$$\eta_{[l_1,l_2]} = \sum_{l=l_1}^{l_2} \eta_l \text{ and } \eta_{\leq l_2} = \sum_{l=-\infty}^{l_2} \eta_l.$$

For $l \in \mathbb{Z}$ let $I_l = \{\xi \in \mathbb{R} : |\xi| \in [2^{l-1}, 2^{l+1}]\}$. For $l \in \mathbb{Z}_+$ let $\widetilde{I}_l = [-2,2]$ if $l = 0$ and $\widetilde{I}_l = I_l$ if $l \geq 1$. For $k \in \mathbb{Z}$ and $j \in \mathbb{Z}_+$ let

$$\begin{cases} D_{k,j} = \{(\xi,\tau) \in \mathbb{R}\times\mathbb{R} : \xi \in I_k,\ \tau - \omega(\xi) \in \widetilde{I}_j\} \text{ if } k \geq 1; \\ D_{k,j} = \{(\xi,\tau) \in \mathbb{R}\times\mathbb{R} : \xi \in I_k,\ \tau \in \widetilde{I}_j\} \text{ if } k \leq 0, \end{cases}$$

where, for $\xi \in \mathbb{R}$,

$$\omega(\xi) = -\xi|\xi|. \tag{2.1}$$

We define first the normed spaces $X_k = X_k(\mathbb{R}\times\mathbb{R})$, $k \in \mathbb{Z}_+$: for $k \geq 1$ let

$$\begin{aligned} X_k =&\{f \in L^2 :\ f \text{ supported in } I_k \times \mathbb{R} \text{ and} \\ &||f||_{X_k} := \sum_{j=0}^{\infty} 2^{j/2}\beta_{k,j}||\eta_j(\tau - \omega(\xi))f(\xi,\tau)\,||_{L^2_{\xi,\tau}} < \infty\}, \end{aligned} \tag{2.2}$$

where

$$\beta_{k,j} = 1 + 2^{(j-2k)/2}. \tag{2.3}$$

The precise choice of the coefficients $\beta_{k,j}$ is important in order for all the bilinear estimates (3.4), (3.5), (3.6), and (3.7) to hold (see the discussion in section 5). Notice that $2^{j/2}\beta_{k,j} \approx 2^j$ when k is small. For $k = 0$ we define

$$\begin{aligned} X_0 =& \{f \in L^2 : \ f \text{ supported in } \widetilde{I}_0 \times \mathbb{R} \text{ and} \\ & ||f||_{X_0} := \sum_{j=0}^{\infty} \sum_{k'=-\infty}^{1} 2^{j-k'/2} ||\eta_j(\tau)\chi_{k'}(\xi) f(\xi,\tau)\,||_{L^2_{\xi,\tau}} < \infty\}. \end{aligned} \tag{2.4}$$

The spaces X_k are not sufficient for our purpose, due to various logarithmic divergences involving the modulation variable. For $k \geq 100$ and $k = 0$ we also define the normed spaces $Y_k = Y_k(\mathbb{R} \times \mathbb{R})$. For $k \geq 100$ we define

$$\begin{aligned} Y_k = \{ & f \in L^2 : f \text{ supported in } \bigcup_{j=0}^{k-1} D_{k,j} \text{ and} \\ & ||f||_{Y_k} := 2^{-k/2} ||\mathcal{F}_{(2)}^{-1}[(\tau - \omega(\xi) + i) f(\xi,\tau)]||_{L^1_x L^2_t} < \infty\}. \end{aligned} \tag{2.5}$$

For $k = 0$ we define

$$\begin{aligned} Y_0 = \{ & f \in L^2 : f \text{ supported in } \widetilde{I}_0 \times \mathbb{R} \text{ and} \\ & ||f||_{Y_0} := \sum_{j=0}^{\infty} 2^j ||\mathcal{F}_{(2)}^{-1}[\eta_j(\tau) f(\xi,\tau)]||_{L^1_x L^2_t} < \infty\}. \end{aligned} \tag{2.6}$$

Then we define

$$Z_k := X_k \text{ if } 1 \leq k \leq 99 \text{ and } Z_k := X_k + Y_k \text{ if } k \geq 100 \text{ or } k = 0. \tag{2.7}$$

The spaces Z_k are our basic normed spaces. The spaces X_k are $X^{s,b}$-type spaces; the spaces Y_k are relevant due to the local smoothing inequality

$$||\partial_x u||_{L^\infty_x L^2_t} \leq C||(\partial_t + \mathcal{H}\partial_x^2)u||_{L^1_x L^2_t} \text{ for any } u \in \mathcal{S}(\mathbb{R} \times \mathbb{R}).$$

For $k \in \mathbb{Z}_+$ let

$$\begin{cases} A_k(\xi,\tau) = \tau - \omega(\xi) + i \text{ if } k \geq 1; \\ A_k(\xi,\tau) = \tau + i \text{ if } k = 0. \end{cases}$$

For $\sigma \geq 0$ we define the normed spaces $F^\sigma = F^\sigma(\mathbb{R} \times \mathbb{R})$ and $N^\sigma = N^\sigma(\mathbb{R} \times \mathbb{R})$:

$$F^\sigma = \Big\{ u \in C(\mathbb{R} : \widetilde{H}^\infty) : ||u||^2_{F^\sigma} := \sum_{k=0}^{\infty} 2^{2\sigma k} ||\eta_k(\xi)(I - \partial_\tau^2)\mathcal{F}_{(2)}(u)||^2_{Z_k} < \infty \Big\}, \tag{2.8}$$

and

$$N^\sigma = \Big\{ u \in C(\mathbb{R} : \widetilde{H}^\infty) : ||u||_{N^\sigma}^2 := \sum_{k=0}^{\infty} 2^{2\sigma k} ||\eta_k(\xi) A_k(\xi,\tau)^{-1} \mathcal{F}_{(2)}(u)||_{Z_k}^2 < \infty \Big\}. \tag{2.9}$$

We summarize now some basic properties of the spaces Z_k. Using the definitions, if $k \geq 1$ and $f_k \in Z_k$ then f_k can be written in the form

$$\begin{cases} f_k = \sum_{j=0}^{\infty} f_{k,j} + g_k; \\ \sum_{j=0}^{\infty} 2^{j/2} \beta_{k,j} ||f_{k,j}||_{L^2} + ||g_k||_{Y_k} \leq 2||f_k||_{Z_k}, \end{cases} \tag{2.10}$$

such that $f_{k,j}$ is supported in $D_{k,j}$ and g_k is supported in $\bigcup_{j=0}^{k-1} D_{k,j}$ (if $k \leq 99$ then $g_k \equiv 0$). If $f_0 \in Z_0$ then f_0 can be written in the form

$$\begin{cases} f_0 = \sum_{j=0}^{\infty} \sum_{k'=-\infty}^{1} f_{0,j}^{k'} + \sum_{j=0}^{\infty} g_{0,j}; \\ \sum_{j=0}^{\infty} \sum_{k'=-\infty}^{1} 2^{j-k'/2} ||f_{0,j}^{k'}||_{L^2} + \sum_{j=0}^{\infty} 2^j ||\mathcal{F}_{(2)}^{-1}(g_{0,j})||_{L_x^1 L_t^2} \leq 2||f_0||_{Z_0}, \end{cases} \tag{2.11}$$

such that $f_{0,j}^{k'}$ is supported in $D_{k',j}$ and $g_{0,j}$ is supported in $\widetilde{I}_0 \times \widetilde{I}_j$. The main properties of the spaces Z_k are listed in Lemma 2.1 below (see [7, Section 4] for complete proofs).

Lemma 2.1. *(a) If $m, m' : \mathbb{R} \to \mathbb{C}$, $k \geq 0$, and $f_k \in Z_k$ then*

$$\begin{cases} ||m(\xi) f_k(\xi,\tau)||_{Z_k} \leq C||\mathcal{F}_{(1)}^{-1}(m)||_{L^1(\mathbb{R})} ||f_k||_{Z_k}; \\ ||m'(\tau) f_k(\xi,\tau)||_{Z_k} \leq C||m'||_{L^\infty(\mathbb{R})} ||f_k||_{Z_k}. \end{cases} \tag{2.12}$$

(b) If $k \geq 1$, $j \geq 0$, and $f_k \in Z_k$ then

$$||\eta_j(\tau - \omega(\xi)) f_k(\xi,\tau)||_{X_k} \leq C||f_k||_{Z_k}. \tag{2.13}$$

(c) If $k \geq 1$, $j \in [0,k]$, and f_k is supported in $I_k \times \mathbb{R}$ then

$$||\mathcal{F}_{(2)}^{-1}[\eta_{\leq j}(\tau - \omega(\xi)) f_k(\xi,\tau)]||_{L_x^1 L_t^2} \leq C||\mathcal{F}_{(2)}^{-1}(f_k)||_{L_x^1 L_t^2}. \tag{2.14}$$

(d) If $k \geq 0$, $t \in \mathbb{R}$, and $f_k \in Z_k$ then

$$\begin{cases} \big|\big| \int_{\mathbb{R}} f_k(\xi,\tau) e^{it\tau}\, d\tau \big|\big|_{L_\xi^2} \leq C||f_k||_{Z_k} \text{ if } k \geq 1; \\ \big|\big| \int_{\mathbb{R}} f_0(\xi,\tau) e^{it\tau}\, d\tau \big|\big|_{B_0} \leq C||f_0||_{Z_0} \text{ if } k = 0. \end{cases} \tag{2.15}$$

As a consequence,

$$\sup_{t \in \mathbb{R}} \|u(.,t)\|_{\widetilde{H}^\sigma} \leq C_\sigma \|u\|_{F^\sigma} \text{ for any } \sigma \geq 0 \text{ and } u \in F^\sigma. \tag{2.16}$$

(e) (maximal function estimate) If $k \geq 1$ and $(I - \partial_\tau^2) f_k \in Z_k$ then

$$||\mathcal{F}_{(2)}^{-1}(f_k)||_{L_x^2 L_t^\infty} \leq C 2^{k/2} ||(I - \partial_\tau^2) f_k||_{Z_k}. \tag{2.17}$$

(f) (local smoothing estimate) If $k \geq 1$ and $f_k \in Z_k$ then

$$||\mathcal{F}_{(2)}^{-1}(f_k)||_{L_x^\infty L_t^2} \leq C 2^{-k/2} ||f_k||_{Z_k}. \tag{2.18}$$

3. Linear and bilinear estimates

In this section we state our main linear and bilinear estimates. For any $u \in C(\mathbb{R} : L^2)$ let $\widetilde{u}(.,t) \in C(\mathbb{R} : L^2)$ denote its partial Fourier transform with respect to the variable x. For $\phi \in L^2(\mathbb{R})$ let $W(t)\phi \in C(\mathbb{R} : L^2)$ denote the solution of the free Benjamin–Ono evolution given by

$$[W(t)\phi]^{\sim}(\xi, t) = e^{it\omega(\xi)} \mathcal{F}_{(1)}(\phi)(\xi), \tag{3.1}$$

where $\omega(\xi)$ is defined in (2.1). Assume $\psi : \mathbb{R} \to [0,1]$ is an even smooth function supported in the interval $[-8/5, 8/5]$ and equal to 1 in the interval $[-5/4, 5/4]$. Propositions 3.1 and 3.2 below are our main linear estimates (see [7, Section 5] for proofs).

Proposition 3.1. *If $\sigma \geq 0$ and $\phi \in \widetilde{H}^\infty$ then*

$$||\psi(t) \cdot (W(t)\phi)||_{F^\sigma} \leq C_\sigma ||\phi||_{\widetilde{H}^\sigma}. \tag{3.2}$$

Proposition 3.2. *If $\sigma \geq 0$ and $u \in N^\sigma$ then*

$$\Big|\Big|\psi(t) \cdot \int_0^t W(t-s)(u(s))\, ds\Big|\Big|_{F^\sigma} \leq C_\sigma ||u||_{N^\sigma}. \tag{3.3}$$

We state now our main dyadic bilinear estimates:

Lemma 3.3. *Assume $k \geq 20$, $k_2 \in [k-2, k+2]$, $f_{k_2} \in Z_{k_2}$, and $f_0 \in Z_0$. Then*

$$2^k \big|\big|\eta_k(\xi) \cdot (\tau - \omega(\xi) + i)^{-1} f_{k_2} * f_0\big|\big|_{Z_k} \leq C ||f_{k_2}||_{Z_{k_2}} ||f_0||_{Z_0}. \tag{3.4}$$

Lemma 3.4. *Assume $k \geq 20$, $k_2 \in [k-2, k+2]$, $f_{k_2} \in Z_{k_2}$, and $f_{k_1} \in Z_{k_1}$ for any $k_1 \in [1, k-10] \cap \mathbb{Z}$. Then*

$$2^k \big|\big|\eta_k(\xi)(\tau - \omega(\xi) + i)^{-1} f_{k_2} * \sum_{k_1=1}^{k-10} f_{k_1}\big|\big|_{Z_k} \leq C ||f_{k_2}||_{Z_{k_2}} \sup_{k_1 \in [1, k-10]} ||(I - \partial_\tau^2) f_{k_1}||_{Z_{k_1}}. \tag{3.5}$$

Lemma 3.5. *Assume $k, k_1, k_2 \in \mathbb{Z}_+$ have the property that $\max(k, k_1, k_2) \leq \min(k, k_1, k_2) + 30$, $f_{k_1} \in Z_{k_1}$, and $f_{k_2} \in Z_{k_2}$. Then*

$$\big|\big|\xi \cdot \eta_k(\xi) \cdot A_k(\xi, \tau)^{-1} f_{k_1} * f_{k_2}\big|\big|_{X_k} \leq C ||f_{k_1}||_{Z_{k_1}} ||f_{k_2}||_{Z_{k_2}}. \tag{3.6}$$

Lemma 3.6. *Assume $k, k_1, k_2 \in \mathbb{Z}_+$, $k_1, k_2 \geq k+10$, $|k_1 - k_2| \leq 2$, $f_{k_1} \in Z_{k_1}$, and $f_{k_2} \in Z_{k_2}$. Then*

$$\big|\big|\xi \cdot \eta_k(\xi) \cdot A_k(\xi,\tau)^{-1} f_{k_1} * f_{k_2}\big|\big|_{X_k} \leq C2^{-k/4} ||f_{k_1}||_{Z_{k_1}} ||f_{k_2}||_{Z_{k_2}}. \tag{3.7}$$

Lemmas 3.4, 3.5, and 3.6 are already proved in [7, Sections 7 and 8] (for Lemma 3.5 see also the bound (8.9) in [7]). We only provide a proof of Lemma 3.3. The main ingredient is Lemma 3.7 below, which follows from Lemma 7.3 in [7].

Lemma 3.7. *Assume that $k \geq 20$, $k_1 \in (-\infty, 1] \cap \mathbb{Z}$, $k_2 \in [k-2, k+2]$, $j, j_1, j_2 \in \mathbb{Z}_+$, f_{k_1,j_1} is an L^2 function supported in D_{k_1,j_1}, and f_{k_2,j_2} is an L^2 function supported in D_{k_2,j_2}. Then*

$$\begin{aligned} &2^k 2^{j/2} \beta_{k,j} ||\eta_k(\xi)\eta_j(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1}(f_{k_1,j_1} * f_{k_2,j_2})||_{L^2} \\ &\qquad \leq C(2^{k_1/2} + 2^{-k/2})^{-1} \cdot 2^{j_1} ||f_{k_1,j_1}||_{L^2} \cdot 2^{j_2/2} \beta_{k_2,j_2} ||f_{k_2,j_2}||_{L^2}. \end{aligned} \tag{3.8}$$

If $j_1 \geq k + k_1 - 20$ then we have the stronger bound

$$\begin{aligned} &2^k 2^{j/2} \beta_{k,j} ||\eta_k(\xi)\eta_j(\tau - \omega(\xi))(\tau - \omega(\xi) + i)^{-1}(f_{k_1,j_1} * f_{k_2,j_2})||_{L^2} \\ &\quad \leq C2^{-\max(j,j_2)/2}(2^{k_1/2} + 2^{-k/2})^{-1} \cdot 2^{j_1} ||f_{k_1,j_1}||_{L^2} \cdot 2^{j_2/2} \beta_{k_2,j_2} ||f_{k_2,j_2}||_{L^2}. \end{aligned} \tag{3.9}$$

In addition, $\mathbf{1}_{D_{k,j}}(\xi,\tau)(f_{k_1,j_1} * f_{k_2,j_2}) \equiv 0$ unless

$$\begin{cases} \max(j, j_1, j_2) \in [k + k_1 - 10, k + k_1 + 10] \ \textit{or} \\ \max(j, j_1, j_2) \geq k + k_1 + 10 \ \textit{and} \max(j, j_1, j_2) - \operatorname{med}(j, j_1, j_2) \leq 10. \end{cases} \tag{3.10}$$

The restriction (3.10) follows from the elementary dispersive identity

$$|\omega(\xi_1 + \xi_2) - \omega(\xi_1) - \omega(\xi_2)| = 2\min(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|) \cdot \operatorname{med}(|\xi_1|, |\xi_2|, |\xi_1 + \xi_2|),$$

where $\operatorname{med}(\alpha_1, \alpha_2, \alpha_3) = \alpha_1 + \alpha_2 + \alpha_3 - \max(\alpha_1, \alpha_2, \alpha_3) - \min(\alpha_1, \alpha_2, \alpha_3)$ for any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

Proof of Lemma 3.3. We use the representations (2.10) and (2.11) and analyze three cases.

Case 1: $f_0 = f^{k_1}_{0,j_1}$ is supported in D_{k_1,j_1}, $f_{k_2} = f_{k_2,j_2}$ is supported in D_{k_2,j_2}, $j_1, j_2 \geq 0$, $k_1 \leq 1$, $||f_0||_{Z_0} \approx 2^{j_1 - k_1/2} ||f^{k_1}_{0,j_1}||_{L^2}$, and $||f_{k_2}||_{Z_{k_2}} \approx 2^{j_2/2} \beta_{k_2,j_2} ||f_{k_2,j_2}||_{L^2}$. The bound (3.4) which we have to prove becomes

$$2^k \big|\big|\eta_k(\xi) \cdot (\tau - \omega(\xi) + i)^{-1} f_{k_2,j_2} * f^{k_1}_{0,j_1}\big|\big|_{Z_k} \leq C2^{j_1 - k_1/2} ||f^{k_1}_{0,j_1}||_{L^2} \cdot 2^{j_2/2} \beta_{k_2,j_2} ||f_{k_2,j_2}||_{L^2}. \tag{3.11}$$

Let $h_k(\xi,\tau) = \eta_k(\xi)(\tau - \omega(\xi) + i)^{-1}(f_{k_2,j_2} * f^{k_1}_{0,j_1})(\xi,\tau)$. The first observation is that for most choices of j_1 and j_2, depending on k and k_1, the function h_k is supported in a bounded number of regions $D_{k,j}$, so (3.8) suffices to control $2^k ||h_k||_{X_k}$. In

view of (3.10), the function h_k is supported in a bounded number of regions $D_{k,j}$, and (3.11) follows from (3.8), unless

$$\begin{cases} |j_1-(k+k_1)|\leq 10 \text{ and } j_2\leq k+k_1+10 \text{ or}\\ |j_2-(k+k_1)|\leq 10 \text{ and } j_1\leq k+k_1+10 \text{ or}\\ j_1,j_2\geq k+k_1-10 \text{ and } |j_1-j_2|\leq 10.\end{cases} \tag{3.12}$$

Assume (3.12) holds. Using (3.10), $\mathbf{1}_{D_{k,j}}(\xi,\tau)\cdot h_k\equiv 0$ unless $j\leq \max(j_1,j_2)+C$. We have two cases: if $j_1\geq k+k_1-20$, then, in view of (3.12), $j_2\leq j_1+C$ and the function h_k is supported in $\bigcup_{j\leq j_1+C}D_{k,j}$. By (3.9),

$$\begin{aligned}2^k||h_k||_{X_k}&\leq C2^k\sum_{j\leq j_1+C}2^{j/2}\beta_{k,j}||\eta_j(\tau-\omega(\xi))h_k(\xi,\tau)||_{L^2}\\ &\leq C\big[\sum_{j\leq j_1+C}2^{-\max(j,j_2)/2}\big]2^{-k_1/2}\cdot 2^{j_1}||f^{k_1}_{0,j_1}||_{L^2}\cdot 2^{j_2/2}\beta_{k_2,j_2}||f_{k_2,j_2}||_{L^2},\end{aligned}$$

which suffices for (3.11).

Assume now that $j_1\leq k+k_1-20$, so, in view of (3.12), $|j_2-(k+k_1)|\leq 10$ and the function h_k is supported in $\bigcup_{j\leq k+k_1+C}D_{k,j}$. Then, using Lemma 2.1 (b) and (c)

$$\begin{aligned}2^k||h_k||_{Z_k}&\leq C2^{k/2}||\mathcal{F}^{-1}_{(2)}[(\tau-\omega(\xi)+i)h_k(\xi,\tau)]||_{L^1_xL^2_t}\\ &\leq C2^{k/2}||\mathcal{F}^{-1}_{(2)}(f^{k_1}_{0,j_1})||_{L^2_xL^\infty_t}||\mathcal{F}^{-1}_{(2)}(f_{k_2,j_2})||_{L^2_xL^2_t}\\ &\leq C2^{(j_1-k_1)/2}||f^{k_1}_{0,j_1}||_{L^2}\cdot 2^{(k+k_1)/2}||f_{k_2,j_2}||_{L^2},\end{aligned}$$

which suffices for (3.11) since $|j_2-(k+k_1)|\leq 10$.

Case 2: $f_0=f^{k_1}_{0,j_1}$ is supported in D_{k_1,j_1}, $j_1\geq 0$, $k_1\leq 1$, $f_{k_2}=g_{k_2}$ is supported in $\bigcup_{j_2\leq k_2-1}D_{k_2,j_2}$, $||f_0||_{Z_0}\approx 2^{j_1-k_1/2}||f^{k_1}_{0,j_1}||_{L^2}$, and $||f_{k_2}||_{Z_{k_2}}\approx ||g_{k_2}||_{Y_{k_2}}$. The bound (3.4) which we have to prove becomes

$$2^k\big|\big|\eta_k(\xi)\cdot(\tau-\omega(\xi)+i)^{-1}g_{k_2}*f^{k_1}_{0,j_1}\big|\big|_{Z_k}\leq C2^{j_1-k_1/2}||f^{k_1}_{0,j_1}||_{L^2}\cdot||g_{k_2}||_{Y_{k_2}}. \tag{3.13}$$

We have two cases: if $j_1\geq k+k_1-20$ then let $g_{k_2,j_2}(\xi_2,\tau_2)=g_{k_2}(\xi_2,\tau_2)\eta_{j_2}(\tau_2-\omega(\xi_2))$. Using X_k norms, Lemma 2.1 (b), (3.10), and (3.9), the left-hand side of (3.13) is dominated by

$$\begin{aligned}&C\sum_{j,j_2\geq 0}2^k2^{j/2}\beta_{k,j}||\eta_k(\xi)\eta_j(\tau-\omega(\xi))(\tau-\omega(\xi)+i)^{-1}(f^{k_1}_{0,j_1}*g_{k_2,j_2})||_{L^2}\\ &\leq C(2^{k_1/2}+2^{-k/2})^{-1}\cdot 2^{j_1}||f^{k_1}_{0,j_1}||_{L^2}\sum_{j,j_2\geq 0}2^{-\max(j,j_2)/2}\cdot 2^{j_2/2}\beta_{k_2,j_2}||g_{k_2,j_2}||_{L^2}\\ &\leq C(2^{k_1/2}+2^{-k/2})^{-1}\cdot 2^{j_1}||f^{k_1}_{0,j_1}||_{L^2}\cdot||g_{k_2}||_{Y_{k_2}},\end{aligned}$$

which suffices to prove (3.13) in this case.

Assume now that $j_1 \leq k+k_1-20$. Let

$$\begin{cases} g_{k_2,\mathrm{low}}(\xi_2,\tau_2) = g_{k_2}(\xi_2,\tau_2)\cdot \eta_{\leq k+k_1-20}(\tau_2-\omega(\xi_2)); \\ g_{k_2,\mathrm{high}}(\xi_2,\tau_2) = g_{k_2}(\xi_2,\tau_2)\cdot (1-\eta_{\leq k+k_1-20}(\tau_2-\omega(\xi_2))). \end{cases}$$

In view of (3.10), the function $f_{0,j_1}^{k_1} * g_{k_2,\mathrm{low}}$ is supported in the union of a bounded number of dyadic regions $D_{k,j}$, $|j-(k+k_1)|\leq C$. Then, using X_k norms in the left-hand side of (3.13) and Lemma 2.1 (c) and (f),

$$\begin{aligned} 2^k\big|\big|\eta_k(\xi)\cdot(\tau-\omega(\xi)+i)^{-1}g_{k_2,\mathrm{low}} * f_{0,j_1}^{k_1}\big|\big|_{Z_k} &\leq C2^k 2^{-(k+k_1)/2}||f_{0,j_1}^{k_1} * g_{k_2,\mathrm{low}}||_{L^2} \\ &\leq C2^{(k-k_1)/2}||\mathcal{F}^{-1}_{(2)}(f_{0,j_1}^{k_1})||_{L^2_xL^\infty_t}||\mathcal{F}^{-1}_{(2)}(g_{k_2,\mathrm{low}})||_{L^\infty_xL^2_t} \\ &\leq C2^{(k-k_1)/2}\cdot 2^{j_1/2}||f_{0,j_1}^{k_1}||_{L^2}\cdot 2^{-k/2}||g_{k_2,\mathrm{low}}||_{Y_{k_2}} \\ &\leq C2^{(j_1-k_1)/2}||f_{0,j_1}^{k_1}||_{L^2}\cdot ||g_{k_2}||_{Y_{k_2}}, \end{aligned}$$

which agrees with (3.13). To handle the part corresponding to $f_{0,j_1}^{k_1} * g_{k_2,\mathrm{high}}$, we notice that, in view of Lemma 2.1 (b),

$$\|g_{k_2,\mathrm{high}}\|_{L^2_{x,t}} \leq C2^{-(k+k_1)/2}||g_{k_2}||_{Y_{k_2}}.$$

Then, using Y_k norms in the left-hand side of (3.13) and Lemma 2.1 (c),

$$\begin{aligned} 2^k\big|\big|\eta_k(\xi)\cdot(\tau-\omega(\xi)+i)^{-1}g_{k_2,\mathrm{high}} * f_{0,j_1}^{k_1}\big|\big|_{Z_k} &\leq C2^{k/2}||\mathcal{F}^{-1}_{(2)}(f_{0,j_1}^{k_1} * g_{k_2,\mathrm{high}})||_{L^1_xL^2_t} \\ &\leq C2^{k/2}||\mathcal{F}^{-1}_{(2)}(f_{0,j_1}^{k_1})||_{L^2_xL^\infty_t}||\mathcal{F}^{-1}_{(2)}(g_{k_2,\mathrm{high}})||_{L^2} \\ &\leq C2^{k/2}\cdot 2^{j_1/2}||f_{0,j_1}^{k_1}||_{L^2}\cdot 2^{-(k+k_1)/2}||g_{k_2}||_{Y_{k_2}} \\ &\leq C2^{(j_1-k_1)/2}||f_{0,j_1}^{k_1}||_{L^2}\cdot ||g_{k_2}||_{Y_{k_2}}, \end{aligned}$$

which completes the proof of (3.13).

Case 3: $f_0 = g_{0,j_1}$ is supported in $\widetilde{I}_0\times\widetilde{I}_{j_1}$, $j_1\geq 0$, $||f_0||_{Z_0}\approx 2^{j_1}||\mathcal{F}^{-1}_{(2)}(g_{0,j_1})||_{L^1_xL^2_t}$. The bound (3.4) which we have to prove becomes

$$2^k\big|\big|\eta_k(\xi)\cdot(\tau-\omega(\xi)+i)^{-1}f_{k_2} * g_{0,j_1}\big|\big|_{Z_k} \leq C2^{j_1}||\mathcal{F}^{-1}(g_{0,j_1})||_{L^1_xL^2_t}\cdot||f_{k_2}||_{Z_{k_2}}. \quad (3.14)$$

This is proved in [7, Estimate (7.12)], which completes the proof of Lemma 3.3. □

We prove now our main bilinear estimate for functions in F^σ.

Proposition 3.8. *If $\sigma\geq 0$ and $u,v\in F^\sigma$ then*

$$||\partial_x(uv)||_{N^\sigma} \leq C_\sigma(||u||_{F^\sigma}||v||_{F^0}+||u||_{F^0}||v||_{F^\sigma}). \quad (3.15)$$

Proof of Proposition 3.8. For $k\in\mathbb{Z}_+$ we define $F_k(\xi,\tau)=\eta_k(\xi)\mathcal{F}_{(2)}(u)(\xi,\tau)$ and $G_k(\xi,\tau)=\eta_k(\xi)\mathcal{F}_{(2)}(v)(\xi,\tau)$. Then

$$\begin{cases} ||u||^2_{F^\sigma} = \sum_{k_1=0}^\infty 2^{2\sigma k_1}||(I-\partial_\tau^2)F_{k_1}||^2_{Z_{k_1}}; \\ ||v||^2_{F^\sigma} = \sum_{k_2=0}^\infty 2^{2\sigma k_2}||(I-\partial_\tau^2)G_{k_2}||^2_{Z_{k_2}}, \end{cases}$$

and

$$\eta_k(\xi)\mathcal{F}[\partial_x(u\cdot v)](\xi,\tau)=C\xi\sum_{k_1,k_2\in\mathbb{Z}_+}\eta_k(\xi)[F_{k_1}*G_{k_2}](\xi,\tau).$$

We observe that $\eta_k(\xi)[F_{k_1}*G_{k_2}](\xi,\tau)\equiv 0$ unless

$$\begin{cases} k_1\leq k-10 \text{ and } k_2\in[k-2,k+2] \text{ or}\\ k_1\in[k-2,k+2] \text{ and } k_1\leq k-10 \text{ or}\\ k_1,k_2\in[k-10,k+20] \text{ or}\\ k_1,k_2\geq k+10 \text{ and } |k_1-k_2|\leq 2.\end{cases}$$

For $k,k_1,k_2\in\mathbb{Z}_+$ let

$$H_{k,k_1,k_2}(\xi,\tau)=\eta_k(\xi)A_k(\xi,\tau)^{-1}\xi\cdot(F_{k_1}*G_{k_2})(\xi,\tau).$$

Using the definitions,

$$||\partial_x(u\cdot v)||_{N^\sigma}^2=C\sum_{k\geq 0}2^{2\sigma k}\Big|\Big|\sum_{k_1,k_2}H_{k,k_1,k_2}\Big|\Big|_{Z_k}^2. \tag{3.16}$$

For $k\in\mathbb{Z}_+$ fixed we estimate, using Lemmas 3.3, 3.4, 3.5, and 3.6,

$$\begin{aligned}\Big|\Big|\sum_{k_1,k_2}H_{k,k_1,k_2}\Big|\Big|_{Z_k}\leq&\sum_{|k_2-k|\leq 2}\Big|\Big|\sum_{k_1\leq k-10}H_{k,k_1,k_2}\Big|\Big|_{Z_k}+\sum_{|k_1-k|\leq 2}\Big|\Big|\sum_{k_2\leq k-10}H_{k,k_1,k_2}\Big|\Big|_{Z_k}\\&+\sum_{k_1,k_2\in[k-10,k+20]}||H_{k,k_1,k_2}||_{Z_k}+\sum_{k_1,k_2\geq k+10,\,|k_1-k_2|\leq 2}||H_{k,k_1,k_2}||_{Z_k}\\ \leq& C\Big[\sum_{|k_2-k|\leq 2}||G_{k_2}||_{Z_{k_2}}\Big]\cdot||u||_{F^0}+C\Big[\sum_{|k_1-k|\leq 2}||F_{k_1}||_{Z_{k_1}}\Big]\cdot||v||_{F^0}\\&+C\Big[\sum_{|k_1-k|\leq 20}||F_{k_1}||_{Z_{k_1}}\Big]\Big[\sum_{|k_2-k|\leq 20}||G_{k_2}||_{Z_{k_2}}\Big]\\&+C2^{-k/4}\Big[\sum_{k_1\geq k}||F_{k_1}||_{Z_{k_1}}^2\Big]^{1/2}\Big[\sum_{k_2\geq k}||G_{k_2}||_{Z_{k_2}}^2\Big]^{1/2}.\end{aligned}$$

The bound (3.15) follows. □

4. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. The main ingredients are Propositions 3.1, 3.2, and 3.8, and the bound (2.16). For any interval $I=[t_0-a,t_0+a]$, $t_0\in\mathbb{R}$, $a\in[0,5/4]$, and $\sigma\geq 0$ we define the normed space

$$F^\sigma(I)=\{u\in C(I:\widetilde{H}^\infty):||u||_{F^\sigma(I)}=\inf_{\widetilde{u}\equiv u\text{ on }\mathbb{R}\times I}||\widetilde{u}||_{F^\sigma}<\infty\}.$$

With this notation, the estimate in Proposition 3.1 becomes

$$||W(t-t_0)\phi||_{F^\sigma([t_0-a,t_0+a])}\leq C_\sigma||\phi||_{\widetilde{H}^\sigma}\text{ for any }\phi\in\widetilde{H}^\infty. \tag{4.1}$$

By combining Propositions 3.2 and 3.8 we obtain (with $I = [t_0 - a, t_0 + a]$)

$$\Big|\Big| \int_{t_0}^{t} W(t-s)(\partial_x(u \cdot v)(s))\,ds \Big|\Big|_{F^\sigma(I)} \leq C_\sigma(||u||_{F^\sigma(I)}||v||_{F^0(I)} + ||u||_{F^0(I)}||v||_{F^\sigma(I)}), \tag{4.2}$$

for any $u, v \in F^\sigma([t_0 - a, t_0 + a])$, $\sigma \geq 0$. Finally, the estimate (2.16) becomes

$$\sup_{t \in [t_0 - a, t_0 + a]} \|u(.,t)\|_{\widetilde{H}^\sigma} \leq C_\sigma \|u\|_{F^\sigma([t_0 - a, t_0 + a])} \text{ for any } u \in F^\sigma([t_0 - a, t_0 + a]). \tag{4.3}$$

Proof of existence. We prove first the existence part of Theorem 1.1, including the Lipschitz bounds in (b) and (c). Assume, as in Theorem 1.1, that $\phi \in \widetilde{H}^\infty \cap B(\overline{\epsilon}, \widetilde{H}^0)$. We define recursively $u_n \in C([-1,1] : \widetilde{H}^\infty)$,

$$\begin{cases} u_0 = W(t)\phi; \\ u_{n+1} = W(t)\phi - \frac{1}{2}\int_0^t W(t-s)(\partial_x(u_n^2)(s))\,ds \text{ for } n \in \mathbb{Z}_+. \end{cases} \tag{4.4}$$

We show first that

$$\|u_n\|_{F^0([-1,1])} \leq C\|\phi\|_{\widetilde{H}^0} \text{ for any } n \in \mathbb{Z}_+ \text{ if } \overline{\epsilon} \text{ is sufficiently small.} \tag{4.5}$$

The bound (4.5) holds for $n = 0$, due to (4.1). Then, using (4.2) with $\sigma = 0$, it follows that

$$\|u_{n+1}\|_{F^0([-1,1])} \leq C\|\phi\|_{\widetilde{H}^0} + C\|u_n\|^2_{F^0([-1,1])},$$

which leads to (4.5) by induction over n.

We show now that

$$\|u_n - u_{n-1}\|_{F^0([-1,1])} \leq C2^{-n} \cdot \|\phi\|_{\widetilde{H}^0} \text{ for any } n \in \mathbb{Z}_+ \text{ if } \overline{\epsilon} \text{ is sufficiently small.} \tag{4.6}$$

This is clear for $n = 0$ (with $u_{-1} \equiv 0$). Then, using (4.2) with $\sigma = 0$, the definition (4.4), and (4.5)

$$\|u_{n+1} - u_n\|_{F^0([-1,1])} \leq C \cdot \overline{\epsilon} \cdot \|u_n - u_{n-1}\|_{F^0([-1,1])},$$

which leads to (4.6) by induction over n.

We show now that

$$\|u_n\|_{F^\sigma([-1,1])} \leq C(\sigma, \|\phi\|_{\widetilde{H}^\sigma}) \text{ for any } n \in \mathbb{Z}_+ \text{ and } \sigma \in [0, \infty). \tag{4.7}$$

For $\sigma \in [0,2]$, the bound (4.7) follows in the same way as the bound (4.5), by combining (4.1), (4.2), and induction over n. Thus, for (4.7) it suffices to prove that

$$\|J^{\sigma'} u_n\|_{F^{\sigma_0}([-1,1])} \leq C(\sigma', \|J^{\sigma'}\phi\|_{\widetilde{H}^{\sigma_0}}) \text{ for any } n \in \mathbb{Z}_+,\ \sigma' \in \mathbb{Z}_+ \text{ and } \sigma_0 \in [0,1). \tag{4.8}$$

We fix σ_0, and argue by induction over σ'; so we may assume that

$$\|J^{\sigma'-1}(u_n)\|_{F^{\sigma_0}([-1,1])} \leq C(\sigma', \|J^{\sigma'-1}\phi\|_{\widetilde{H}^{\sigma_0}}) \text{ for any } n \in \mathbb{Z}_+, \tag{4.9}$$

and it suffices to prove that

$$\|\partial_x^{\sigma'}(u_n)\|_{F^{\sigma_0}([-1,1])} \leq C(\sigma', \|J^{\sigma'}\phi\|_{\widetilde{H}^{\sigma_0}}) \text{ for any } n \in \mathbb{Z}_+. \tag{4.10}$$

The bound (4.10) for $n = 0$ follows from (4.1). We use the decomposition

$$\partial_x^{\sigma'}\partial_x(u_n^2) = 2\partial_x(\partial_x^{\sigma'}u_n \cdot u_n) + E_n, \tag{4.11}$$

where

$$E_n = \sum_{\sigma_1'+\sigma_2'=\sigma' \text{ and } \sigma_1',\sigma_2'\geq 1} \partial_x(\partial_x^{\sigma_1'}u_n \cdot \partial_x^{\sigma_2'}u_n).$$

Using (4.2) and the induction hypothesis (4.9), we have

$$\Big|\Big| -\frac{1}{2}\int_0^t W(t-s)(E_n(s))\,ds\Big|\Big|_{F^{\sigma_0}([-1,1])} \leq C(\sigma', \|J^{\sigma'-1}\phi\|_{\widetilde{H}^{\sigma_0}}).$$

We use now the definition (4.4), together with the bounds (4.1) and (4.2) to conclude that

$$\begin{aligned}||\partial_x^{\sigma'}u_{n+1}||_{F^{\sigma_0}([-1,1])} &\leq C(\sigma', \|J^{\sigma'}\phi\|_{\widetilde{H}^{\sigma_0}})\\ &+ C\cdot||\partial_x^{\sigma'}u_n||_{F^{\sigma_0}([-1,1])}\cdot||u_n||_{F^0([-1,1])} + C\cdot||\partial_x^{\sigma'}u_n||_{F^0([-1,1])}\cdot||u_n||_{F^{\sigma_0}([-1,1])}.\end{aligned} \tag{4.12}$$

Assume first that $\sigma_0 = 0$: the bound (4.10) follows by induction over n, using (4.5), provided that $\overline{\epsilon}$ is sufficiently small. Thus, for any $\sigma_0 \in [0,1)$, the last term in the right-hand side of (4.12) is also dominated by $C(\sigma', \|J^{\sigma'}\phi\|_{\widetilde{H}^{\sigma_0}})$. Then we use again (4.12) to prove (4.10) for any $\sigma_0 \in [0,1)$.

Finally, we show that

$$\|u_n - u_{n-1}\|_{F^{\sigma}([-1,1])} \leq C(\sigma, \|\phi\|_{\widetilde{H}^{\sigma}})\cdot 2^{-n} \text{ for any } n \in \mathbb{Z}_+ \text{ and } \sigma \in [0,\infty). \tag{4.13}$$

For $\sigma \in [0,2]$, the bound (4.13) follows in the same way as the bound (4.6), by combining (4.1), (4.2), induction over n, and the bounds (4.6) and (4.7). Thus, for (4.13) it suffices to prove that for any $\sigma' \in \mathbb{Z}_+$ and $\sigma_0 \in [0,1)$

$$\|J^{\sigma'}(u_n - u_{n-1})\|_{F^{\sigma_0}([-1,1])} \leq C(\sigma', \|J^{\sigma'}(\phi)\|_{\widetilde{H}^{\sigma_0}})\cdot 2^{-n} \text{ for any } n \in \mathbb{Z}_+. \tag{4.14}$$

As before, we fix σ_0 and argue by induction over σ'. We use the decomposition

$$\partial_x^{\sigma'}(\partial_x(u_n^2 - u_{n-1}^2)) = \partial_x[\partial_x^{\sigma'}(u_n - u_{n-1})\cdot(u_n + u_{n-1})] + \text{ other terms.} \tag{4.15}$$

We use the induction hypothesis and (4.2) to bound the $F^{\sigma_0}([-1,1])$ norm of the other terms in the decomposition above by $C(\sigma', \|J^{\sigma'}(\phi)\|_{\widetilde{H}^{\sigma_0}})\cdot 2^{-n}$. Then, using (4.2) again, we obtain

$$\begin{aligned}||\partial_x^{\sigma'}(u_{n+1} - u_n)||_{F^{\sigma_0}([-1,1])} &\leq C(\sigma', \|J^{\sigma'}\phi\|_{\widetilde{H}^{\sigma_0}})\cdot 2^{-n}\\ &+ C\cdot||\partial_x^{\sigma'}(u_n - u_{n-1})||_{F^{\sigma_0}([-1,1])}\cdot(||u_n||_{F^0([-1,1])} + \|u_{n-1}\|_{F^0([-1,1])})\\ &+ C\cdot||\partial_x^{\sigma'}(u_n - u_{n-1})||_{F^0([-1,1])}\cdot(||u_n||_{F^{\sigma_0}([-1,1])} + ||u_{n-1}||_{F^{\sigma_0}([-1,1])}).\end{aligned} \tag{4.16}$$

As before, we use this inequality first for $\sigma_0 = 0$, together with (4.5) and induction over n, to prove (4.14) for $\sigma_0 = 0$. Then we incorporate the last term in the right-hand side of (4.16) into the first term $C(\sigma', \|J^{\sigma'}\phi\|_{\widetilde{H}^{\sigma_0}}) \cdot 2^{-n}$, and apply the inequality again for any $\sigma_0 \in [0, 1)$. The bound (4.14) follows.

We can now use (4.13) and (4.3) to construct

$$u = \lim_{n\to\infty} u_n \in C([-1, 1] : \widetilde{H}^\infty).$$

In view of (4.4),

$$u = W(t)\phi - \frac{1}{2}\int_0^t W(t-s)(\partial_x(u^2(s)))\,ds \text{ on } \mathbb{R} \times [-1, 1],$$

so $S^\infty(\phi) = u$ is a solution of the initial-value problem (1.1).

For Theorem 1.1 (b) and (c), it suffices to show that if $\sigma \in [0, \infty)$ and $\phi, \phi' \in B(\overline{\epsilon}, \widetilde{H}) \cap H^\infty$ then

$$\sup_{t\in[-1,1]} ||S^\infty(\phi) - S^\infty(\phi')||_{\widetilde{H}^\sigma} \leq C(\sigma, ||\phi||_{\widetilde{H}^\sigma} + \|\phi'\|_{\widetilde{H}^\sigma}) \cdot ||\phi - \phi'||_{\widetilde{H}^\sigma}. \tag{4.17}$$

Part (b) corresponds to the case $\sigma = 0$. To prove (4.17), we define the sequences u_n and u'_n, $n \in \mathbb{Z}_+$, as in (4.4). In view of (4.3), for (4.17) it suffices to prove that for $\sigma \geq 0$ and $n \in \mathbb{Z}_+$

$$\|u_n - u'_n\|_{F^\sigma([-1,1])} \leq C(\sigma, ||\phi||_{\widetilde{H}^\sigma} + \|\phi'\|_{\widetilde{H}^\sigma}) \cdot ||\phi - \phi'||_{\widetilde{H}^\sigma}. \tag{4.18}$$

The proof of (4.18) is similar to the proof of (4.13): for $\sigma \in [0, 2]$ we use (4.1), (4.2), and induction over n. For $\sigma \geq 2$ we write $\sigma = \sigma_0 + \sigma'$, $\sigma' \in \mathbb{Z}_+$, $\sigma_0 \in [0, 1)$, use a decomposition similar to (4.15), and argue by induction over σ'. This completes the proof of (1.5).

Proof of uniqueness. We prove now the uniqueness statement in Theorem 1.1: if $u_1, u_2 \in C([-1, 1] : \widetilde{H}^\infty)$ are solutions of the initial-value problem (1.1) and $u_1(0) = u_2(0) = \phi \in B(\overline{\epsilon}, \widetilde{H}^0) \cap \widetilde{H}^\infty$, then $u_1 \equiv u_2$. For any $T \in [0, 1]$ we define

$$M_l(T) = \|u_l\|_{F^0([-T,T])} \text{ for } l = 1, 2.$$

Since u_l, $l = 1, 2$, are solutions of (1.1), we have

$$u_l(t) = W(t)\phi - \frac{1}{2}\int_0^t W(t-s)(\partial_x(u_l^2(s)))\,ds \text{ on } \mathbb{R} \times [-T, T] \text{ for any } T \in [0, 1]. \tag{4.19}$$

Thus, using (4.1) and (4.2) with $\sigma = 0$,

$$M_l(T) \leq C_0(\overline{\epsilon} + M_l(T)^2),\ l = 1, 2,\ T \in [0, 1],$$

which, provided that $\overline{\epsilon}$ is sufficiently small, implies that

$$M_l(T) \leq 2C_0\overline{\epsilon} \text{ or } M_l(T) \geq (2C_0)^{-1}.$$

The uniqueness statement would follow from (4.19) and (4.2) if we could prove that $M_l(T) \leq 2C_0\overline{\epsilon}$ for all $T \in [0,1]$, $l = 1,2$. For this we need the following quasi-continuity property (see [20, Section 12] for the proof of a similar statement).

Lemma 4.1. *Assume $u \in C([-1,1] : \widetilde{H}^\infty)$ is a solution of the initial-value problem* (1.1) *and define*

$$M(T) = \|u\|_{F^0([-T,T])} \text{ for any } T \in [0,1].$$

Then, for any $\overline{c} > 0$ there is $\overline{C} \geq 1$ (which does not depend on u) such that

$$\limsup_{t\to T} M(t) \leq \overline{C} \cdot \liminf_{t\to T} M(t) + \overline{c}, \tag{4.20}$$

for any $T \in [0,1]$.

5. Two examples

We show first that the bilinear estimate in Lemma 3.4 fails logarithmically if the space Z_k in the left-hand side of (3.5) is replaced with X_k. This is the main reason for using the spaces Y_k.

Proposition 5.1. *Assume $k \geq 20$. Then, for some functions $f_k \in X_k$ and $f_1 \in X_1$,*

$$2^k\big|\big|\eta_k(\xi)(\tau - \omega(\xi) + i)^{-1}(f_k * f_1)\big|\big|_{X_k} \geq C^{-1}k||(I - \partial_\tau^2)f_k||_{X_k}||(I - \partial_\tau^2)f_1||_{X_1}. \tag{5.1}$$

Proof of Proposition 5.1. With ψ as in section 3, let

$$\begin{cases} f_1(\xi_1,\tau_1) = \psi(10(\xi_1 - 2)) \cdot \psi(\tau_1); \\ f_k(\xi_2,\tau_2) = \psi(\xi_2 - 2^k) \cdot \psi(2^{-k-10}(\tau_2 - \omega(\xi_2))). \end{cases}$$

Then $||(I - \partial_\tau^2)f_1||_{X_1} \approx 1$ and $||(I - \partial_\tau^2)f_k||_{X_k} \approx 2^k$. An easy calculation shows that

$$|(f_k * f_1)(\xi,\tau)| \geq C^{-1} \text{ if } \xi \in [2^k - 1/2, 2^k + 1/2] \text{ and } |\tau - \omega(\xi)| \leq 2^k.$$

The bound (5.1) follows from the definitions. □

Our second example justifies the choice of the coefficients $\beta_{k,j}$ in (2.2) and (2.3), as well as the restriction $\sigma \geq 0$ in Proposition 3.8.

Proposition 5.2. *With the definitions of the spaces F^σ and N^σ in* (2.8) *and* (2.9)*, the inequality*

$$||\partial_x(uv)||_{N^\sigma} \leq C_\sigma||u||_{F^\sigma} \cdot ||v||_{F^\sigma} \tag{5.2}$$

holds if and only if $\sigma \geq 0$.

Proof of Proposition 5.2. For k large and $\psi : \mathbb{R} \to [0,1]$ as before, we define u_+ and u_- by

$$\mathcal{F}_{(2)}(u_\pm)(\xi,\tau) = \psi((\xi \mp 2^k)/4) \cdot \psi((\tau - \omega(\xi))/2^{10}).$$

The identity $\omega(\xi_1) + \omega(\xi - \xi_1) = -2^{k+1}\xi + O(1)$ if $|\xi|, |\xi_1 - 2^k| \leq C$ and an easy calculation shows thats

$$\left|\eta_1(\xi)(\tau - \omega(\xi) + i)^{-1} \cdot \mathcal{F}_{(2)}[\partial_x(u_+ \cdot u_-)](\xi,\tau)\right| \geq C^{-1}2^{-k}\eta_1(\xi) \cdot \psi(\tau + 2^{k+1}\xi). \tag{5.3}$$

Then we define v by

$$\mathcal{F}_{(2)}(v)(\xi,\tau) = 2^{-k}\eta_1(\xi) \cdot \mathbf{1}_{[0,\infty)}(\xi) \cdot \psi((\tau + 2^{k+1}\xi)/2^{10})$$

As before, using the identity $2^{k+1}\xi_1 - \omega(\xi - \xi_1) = -\omega(\xi) + O(1)$ if $|\xi_1|, |\xi - 2^k| \leq C$, we compute

$$\left|\eta_k(\xi)(\tau - \omega(\xi) + i)^{-1} \cdot \mathcal{F}_{(2)}[\partial_x(u_+ \cdot v)](\xi,\tau)\right| \geq C^{-1}\psi(\xi - 2^k) \cdot \psi(\tau - \omega(\xi)). \tag{5.4}$$

Using the definitions, we compute easily

$$\|u_\pm\|_{F^\sigma} \approx \|u_\pm\|_{N^\sigma} \approx 2^{\sigma k} \text{ and } \|v\|_{F^\sigma} \approx 2^{-k/2} \cdot \beta_{1,k}.$$

Assuming (5.2), we need $2^{2\sigma k} \geq C^{-1}(2^{-k/2}\beta_{1,k})$ (in view of (5.3)) and $2^{-k/2}\beta_{1,k} \geq C^{-1}$ (in view of (5.4)). This forces $\sigma \geq 0$ and $\beta_{1,k} \approx 2^{k/2}$ (compare with (2.3). □

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